Substructure Synthesis and Its Iterative Improvement for Large Nonconservative Vibratory Systems

A. L. Hale*
University of Illinois at Urbana-Champaign, Urbana, Illinois

A general synthesis method is developed for the dynamic analysis of nonconservative vibratory systems composed of substructures. The idea of the synthesis is to represent each substructure by a reduced-order model and to couple the substructure models together to act as the whole system. For general nonconservative systems, a state space formulation is adopted for each substructure. Reduced-order substructure models are obtained by approximating each state vector as a linear combination of a small number of real trial vectors. The accuracy with which the synthesized model represents the whole system depends on the choices of trial vectors and the number of vectors used. A procedure is developed for increasing the accuracy by iteratively generating improved substructure trial state vectors.

I. Introduction

A SPECIAL class of linear vibratory systems arises when nonconservative forces such as viscous damping forces, follower forces, circulatory forces, and/or aerodynamic forces act on an elastic structure. The vibration analysis of such systems requires determining the eigenvalues and associated eigenvectors, particularly those corresponding to the lower natural modes of vibration. Once an eigensolution is determined, the real parts of the eigenvalues indicate the stability of the system. The eigenvectors can be used to represent the motion of the system; they can also be used in designing an appropriate active control system.

Computational difficulties arise from the very large number of degrees of freedom, perhaps reaching into the tens of thousands, inherent in an accurate finite element model for a complicated structural system. Obtaining a complete eigensolution is almost impossible. For conservative nongyroscopic systems, a direct iterative method capable of producing a partial eigensolution has been developed.[1-4] The method is known as subspace iteration. The eigenproblem is symmetric, the eigenvalues and eigenvectors are real quantities, and subspace iteration produces the real eigensolution by iteratively improving real trial vectors. For nonconservative systems, in addition to the large number of degrees of freedom, the eigenproblem is in terms of asymmetric matrices and its eigenvalues and eigenvectors are complex quantities. Nevertheless, methods similar to subspace iteration can be used for the asymmetric eigenproblem. They produce a complex eigensolution by iteratively improving either conjugate pairs of complex trial vectors[5] or pairs of real trial vectors.[6,7] The large number of degrees of freedom of the mathematical model can, however, be overwhelming. This is particularly true if viscous damping and/or gyroscopic forces are present because they generally lead to an eigenproblem in terms of asymmetric coefficient matrices of twice the dimension as the original matrices.[4] Clearly, a method for prior reduction of the number of degrees of freedom is desirable.

Complicated structural systems can often be modeled by breaking the system into a number of simpler components or substructures. The substructures are modeled separately and then the separate models are coupled together to form the whole structure model. The technique is known as substructure synthesis and its origin can be found in Refs. 8 and 9. The idea of Refs. 8 and 9 is to represent the motion of each substructure by a reduced number of lower substructure modes, i.e., eigenvectors obtained by solving an eigenproblem for each substructure.

References 8 and 9 are concerned with nongyroscopic structural systems that are lightly damped but otherwise conservative, and they take the substructure modes as eigenvectors of the undamped substructures. Many other papers ignore the damping entirely and deal with conservative systems (see the reviews in Refs. 10 and 11). On the other hand, Ref. 12 introduced the idea of using complex substructure eigenvectors to represent each substructure of a general nongyroscopic nonconservative system. There the substructures are coupled together to form a whole structure model before introducing substructure eigenvectors, a process which is often undesirable. As a remedy, Ref. 13 presented a general coupling procedure for the reduced substructure models. Reference 14 also considers representing each nongyroscopic substructure by its complex eigenvectors. References 12-14 emphasize using complex substructure eigenvectors, however, where their use leads to assembled system equations in terms of complex matrices. For complicated substructure models, obtaining the complex eigenvectors can itself be a difficult task. Also, a method that avoids complex matrices is preferable.

This paper develops a *general* substructure synthesis method for nonconservative vibratory systems. Viscous damping and/or gyroscopic forces are included by adopting a state space formulation for each substructure. The state vector contains the displacement and the velocity vectors. Substructures' state equations as well as costate equations are derived as the stationarity condition of a suitable bilinear functional. The substructure equations are in terms of one symmetric matrix and one asymmetric matrix, both of twice the dimension as the original coefficient matrices. To reduce the number of substructure degrees of freedom, each substructure state (costate) vector is represented as the sum of a given number of trial vectors multiplying time-dependent generalized states (costates). The trial vectors are taken to be real vectors and the assembled system equations are in terms of real coefficient matrices. As a method of selecting intitial trial vectors, some trial vectors are taken for the displacement vector only, i.e., those entries corresponding to the velocity vector are equal to zero, while other trial vectors are taken for the velocity only, i.e., those entries corresponding to the displacement vector are equal to zero. Finally, a general

Presented as Paper 82-0772 at the AIAA/ASME/ASCE/AHA 23rd Structures, Structural Dynamics and Materials Conference, New Orleans, La., May 10-12, 1983; submitted May 12, 1982; revision received April 9, 1983.

*Assistant Professor, Aeronautical and Astronautical Engineering Department: appointed jointly to the Civil Engineering Department. Member AIAA.

coupling process permits satisfying the internal boundary compatibility conditions.

The initial synthesis yields only an approximate model for the actual system. One way to improve the accuracy of the model is to simply increase the number of substructure trial vectors. Another possibility is to improve the fixed number of trial vectors. For conservative nongyroscopic systems, Ref. 15 developed a procedure for systematically improving a fixed number of substructure trial vectors until a desired accuracy is obtained. A similar procedure is developed herein for general nonconservative systems composed of substructures. The procedure extends a variant of subspace iteration, developed in Ref. 7 for asymmetric single structure eigenproblems, to structures composed of substructures. Using the procedure, an actual eigensolution consisting of complex eigenvalues and eigenvectors can be obtained for the assembled structure.

II. Substructure Equations

Because many types of nonconservative forces exist, e.g., viscous damping forces, follower forces, circulatory forces, and aerodynamic forces, the focus here is on very general vibratory linear systems. The general setting also permits gyroscopic systems to be considered, either in the presence or in the absence of nonconservative forces. Simplified cases can arise when various forces are not present in one, several, or all of the substructures. In some cases, modification of the basic ideas may be advisable. This task is left to the reader.

Assuming that the whole structure is divided into m substructures, the motion of each substructure s ($s=1,2,..,m$) satisfies the n_s simultaneous differential equations

$$m_s\ddot{u}_s + (g_s + c_{vs})\dot{u}_s + k_{cs}u_s = \eta_s \tag{1}$$

where m_s, g_s, c_{vs}, and k_{cs} are $n_s \times n_s$ real matrices, u_s is the n_s-dimensional vector of generalized displacements, and η_s is an n_s-dimensional vector representing all forces acting on substructure s. The mass matrix m_s is symmetric and positive definite, the gyroscopic matrix g_s is skew symmetric, c_{vs} is the viscous damping matrix, and the symmetric stiffness matrix k_{cs} is either positive definite or only positive semidefinite. The force vector η_s consists of the sum of two parts, namely, a vector η_{Es} of external forces acting on substructure s and a vector η_{Is} of forces exerted on the internal boundaries S_{rs} of substructure s by all adjacent substructures r ($r=1,2,...,m$; $r\neq s$). The vector η_{Es} is assumed to depend explictly on only u_s and $\dot{u}_s$ as follows.

$$\eta_{Es} = \eta_{Es}(u_s, \dot{u}_s) = -k_{Es}u_s - c_{Es}\dot{u}_s \tag{2}$$

where k_{Es} and c_{Es} are general $n_s \times n_s$ real matrices. In view of Eq. (2), Eq. (1) can be written as

$$m_s\ddot{u}_s + c_s\dot{u}_s + k_s u_s = D_s(u_s) = \eta_{Is} \tag{3}$$

where $c_s = g_s + c_{vs} + c_{Es}$ and $k_s = k_{cs} + k_{Es}$ are general matrices and D_s is a matrix differential operator containing first and second derivatives with respect to time. The form assumed in Eq. (2) is general enough to include many common nonconservative forces. Of course, conservative external forces can also contribute to the matrix k_{Es}. Note that instationary external forces such as $\eta_{Es}(u_s, \dot{u}_s, t)$ are not considered.

Dynamical systems represented by equations of motion in the form of Eq. (3) are known as non-selfadjoint systems.[4,16,17] An adjoint problem for each substructure s can be expressed as

$$m_s\ddot{u}_s^* - c_s^T\dot{u}_s^* + k_s^T u_s^* = D_s^*(u_s^*) = \eta_{Is}^* \tag{4}$$

where the symmetry of m_s was recognized and $(\ )^T$ denotes matrix transpositon. The adjoint generalized displacement vector u_s^* and the adjoint internal boundary force vector η_{Is}^* correspond to u_s and η_{Is}, respectively, and D_s^* is the adjoint of the operator D_s. Both Eq. (3) and Eq. (4) can be obtained as the conditions for stationarity, i.e., the conditions for the first variations with respect to u_s and u_s^* to vanish, of the bilinear functional

$$L_s = \int_{t_0}^{t_f} \acute{L}_s \mathrm{d}t, \qquad s=1,2,...,m \tag{5}$$

where the bilinear form $\acute{L}_s$ is

$$\acute{L}_s = u_s^{*T} D_s(u_s) - \eta_{Is}^T u_s^* - \eta_{Is}^{*T} u_s \tag{6}$$

Of course, when taking the first variation of Eq. (5) with respect to u_s, one performs appropriate integrations by parts.

The term $c_s\dot{u}_s$ in Eq. (3) leads to an eigenproblem that is quadratic in the eigenvalue. Therefore, the expansion theorem for the system is in terms of twice as many unknowns as there are generalized coordinates.[18] A number of auxiliary coordinates must be introduced and the n_s second-order equations converted to an equivalent set of $2n_s$ first-order equations. By defining the n_s-dimensional vector of auxiliary coordinates v_s as the generalized velocity vector $\dot{u}_s$, $v_s = \dot{u}_s$ ($s=1,2,...,m$), the first-order *state equations* for substructure s ($s=1,2,...,m$) are obtained by taking variations of the modified bilinear functional

$$L_{ms} = \int_{t_0}^{t_f} \acute{L}_{ms} \mathrm{d}t \tag{7}$$

where

$$\acute{L}_{ms} = u_s^{*T} m_s \dot{v}_s + u_s^{*T} c_s v_s + u_s^{*T} k_s u_s - w_s^T Q_s(v_s - \dot{u}_s) - \eta_{Is}^T u_s^* - \eta_{Is}^{*T} u_s \tag{8}$$

The auxiliary conditon $v_s - \dot{u}_s = \mathbf{0}$ is included as a constraint equation by introducing in Eq. (8) the n_s-dimensional vector $Q_s w_s$ of Lagrange multipliers. The $n_s \times n_s$ matrix Q_s is any positive definite symmetric matrix, where two common choices are $Q_s = m_s$ and $Q_s = I_s$ (I_s is the $n_s \times n_s$ identity matrix). In fact, if $Q_s = I_s$ and $c_s = 0$, then $-w_s$ is the momentum $m_s\dot{u}_s^*$ of the adjoint system. The unknown vector w_s together with the adjoint coordinate vector u_s^* describe the *costate* of each substructure s.

Equation (8) can be written more compactly as

$$\acute{L}_{ms} = y_s^{*T} A_s \dot{y}_s + y_s^{*T} B_s y_s - \gamma_s^T y_s^* - \gamma_s^{*T} y_s \tag{9}$$

where $y_s = \{u_s^T, v_s^T\}^T$ is the state vector, $y_s^* = \{w_s^T, u_s^{*T}\}^T$ is the costate vector, $\gamma_s = \{\mathbf{0}^T, \eta_{Is}^T\}^T$, $\gamma_s^* = \{\eta_{Is}^{*T}, \mathbf{0}^T\}^T$, and

$$A_s = \begin{bmatrix} Q_s & 0 \\ 0 & m_s \end{bmatrix}, \quad B_s = \begin{bmatrix} 0 & -Q_s \\ k_s & c_s \end{bmatrix} \tag{10}$$

Note that A_s is a $2n_s \times 2n_s$ symmetric positive definite matrix and B_s is a $2n_s \times 2n_s$ asymmetric matrix. The conditions for stationarity of the functional (7) are now the following first-order state equations and costate equations, respectively,

$$A_s\dot{y}_s + B_s y_s = \gamma_s \tag{11a}$$

$$-A_s\dot{y}_s^* + B_s^T y_s^* = \gamma_s^* \tag{11b}$$

III. The Coupled Structure

Next, the m substructures are considered to act together so as to form the whole system. The equations of motion for the

assembled system are the conditions for stationarity of the functional

$$L=\sum_{s=1}^{m} L_{ms}=\int_{t_0}^{t_f} \dot{L}\,dt=\int_{t_0}^{t_f}\left(\sum_{s=1}^{m} \dot{L}_{ms}\right)dt \tag{12}$$

where the stationarity conditions are subject to constraints in the form of internal boundary compatibility conditons between adjacent substructures.

Before writing $\dot{L}$ explicitly, it is necessary to recognize that the vectors η_{Is} and η^*_{Is} in Eq. (6) and the corresponding vectors γ_s and γ^*_s in Eq. (9) reflect the effects of adjacent substructures on substructure s, where for each adjacent substructure r and internal boundary S_{rs}

$$(\gamma_s)_{Ir}=\eta_{rs}, \qquad r,s=1,2,\ldots,m;r\neq s \tag{13a}$$

$$(\gamma^*_s)_{Ir}=\eta^*_{rs}, \qquad r,s=1,2,\ldots,m;r\neq s \tag{13b}$$

In Eqs. (13), $(\ )_{Ir}$ denotes the entries in the vectors γ_s and γ^*_s corresponding to the generalized coordinates at the internal boundary points $P\epsilon S_{rs}$, i.e., the corresponding components of $\boldsymbol{u}_s$ and $\boldsymbol{u}^*_s$ at the boundary points P. The vectors η_{rs} and η^*_{rs} are unknown vectors representing the force and the adjoint force, respectively, exerted by substructure r on the internal boundary S_{rs} of substructure s. It is assumed that $(\gamma_s)_{Ir}=\eta_{rs}$ and $(\gamma^*_s)_{Ir}=\eta^*_{rs}$ are m_{rs}-dimensional vectors. Of course, the internal boundary points on S_{rs} are the same as those on S_{sr}, and, therefore, $\eta_{rs}=-\eta_{sr}$, and $\eta^*_{rs}=-\eta^*_{sr}$. The latter statements reflect the fact that the force η_{sr} (adjoint force η^*_{sr}) exerted by substructure s on substructure r must be equal in magnitude and must act in the opposite direction as the force η_{rs} (adjoint force η^*_{rs}), assuming that the coordinate axes for each substructure have the same orientation.

Now, $\dot{L}$ can be written as

$$\dot{L}=\sum_{s=1}^{m}(y_s^{*T}A_s\dot{y}_s+y_s^{*T}B_sy_s)-\sum_{s=1}^{m}\sum_{r=s+1}^{m}\{\eta_{rs}^T[(y_s^*)_{Ir}-(y_r^*)_{Is}]+\eta_{rs}^{*T}[(y_s)_{Ir}-(y_r)_{Is}]\} \tag{14}$$

Equation (14), however, does not yet reflect the geometric compatibility conditions defining an assembled structure. The conditions are $(u_s)_{Ir}-(u_r)_{Is}=0$ and $(u^*_s)_{Ir}-(u^*_r)_{Is}=0$. These conditons can also be written as

$$(y_s)_{Ir}-(y_r)_{Is}=0, \qquad r,s=1,2,\ldots,m;r\neq s \tag{15a}$$

$$(y^*_s)_{Ir}-(y^*_r)_{Is}=0, \qquad r,s=1,2,\ldots,m;r\neq s \tag{15b}$$

where the same notation, $(\ )_{Ir}$, is used to denote the boundary components of u_s and y_s, etc. It is also equivalent to require that $\eta^{*T}_{rs}[(y_s)_{Ir}-(y_r)_{Is}]=0$ and $\eta^T_{rs}[(y^*_s)_{Ir}-(y^*_r)_{Is}]=0$ be satisfied for all possible vectors η^*_{rs} and η_{rs}, respectively. Satisfaction of Eqs. (15) is seen to ensure that the double summation in Eq. (14) equals zero. Note that no compatibility conditions are imposed on the auxiliary coordinates v_s or the Lagrange multipliers w_s. Indeed, v_s and w_s are independent quantities as far as the state and costate equations are concerned (see Ref. 19). Nevertheless, one can also artifically require that $(v_s)_{Ir}-(v_r)_{Is}=0$ and $(w_s)_{Ir}-(w_r)_{Is}=0$ $(r,s=1,2,\ldots,m;\ r\neq s)$. Imposing the latter compatibility conditions is intuitively pleasing, and their satisfaction can be computationally advantageous (see Sec. V).

Next, the number n_s of degrees of freedom for each substructure s $(s=1,2,\ldots m)$ is assumed to large enough that a dramatic reduction in the number of degrees of freedom is needed. The problem is accentuated by working with the $2n_s$ first-order Eq. (11). In the spirit of Galerkin's method, the state vector $y_s(t)$ and the costate vector $y^*_s(t)$ for each substructure s $(s=1,2,\ldots,m)$ and every time t can be represented by the sum of N_s given trial vectors $\psi_{s,i}$ and $\psi^*_{s,i}$ multiplying unknown time-dependent coefficients $a_{si}(t)$ and $a^*_{si}(t)$, respectively, i.e.,

$$y_s(t)=\sum_{i=1}^{N_s}\psi_{s,i}a_{si}(t)=\psi_s a_s(t) \tag{16a}$$

$$y^*_s(t)=\sum_{i=1}^{N_s}\psi^*_{s,i}a^*_{si}(t)=\psi^*_s a^*_s(t) \tag{16b}$$

where ψ_s and ψ^*_s are $2n_s\times N_s$ matrices, $\boldsymbol{a}_s$ and $\boldsymbol{a}^*_s$ are N_s-dimensional vectors, and $N_s\ll n_s$. The trial vectors $\psi_{s,i}$ and $\psi^*_{s,i}$ are called *admissible state vectors* and *admissible costate vectors*, respectively, by analogy with concept of *admissible vectors* discussed in Ref. 11. The vectors $\psi_{s,i}$ $(\psi^*_{s,i})$ must be linearly independent and they must be capable of an accurate representation of the substructure state vector y_s (costate vector y^*_s). The selection of trial vectors $\psi_{s,i}$ and $\psi^*_{s,i}$ is discussed in Sec. V.

Producing a reduced order model of the assembled system requires first that Eq. (16) be substituted into the bilinear form Eq. (14), yielding

$$\dot{L}=\boldsymbol{a}_d^{*T}A_d\dot{\boldsymbol{a}}_d+\boldsymbol{a}_d^{*T}B_d\boldsymbol{a}_d-\eta^TC_d^*\boldsymbol{a}_d^*-\eta^{*T}C_d\boldsymbol{a}_d \tag{17}$$

where $\boldsymbol{a}_d=\{a_1^T,a_2^T,\ldots,a_m^T\}^T$ and $\boldsymbol{a}^*_d=\{a_1^{*T},a_2^{*T},\ldots,a_m^{*T}\}^T$ are the N-dimensional disjoint state vector and disjoint costate vector, respectively,

$$N=\sum_{s=1}^{m}N_s,$$

A_d and B_d are the following $N\times N$ block-diagonal matrices:

$$A_d=\text{block-diag}\quad \psi_s^{*T}A_s\psi_s, \qquad s=1,2,\ldots,m \tag{18a}$$

$$B_d=\text{block-diag}\quad \psi_s^{*T}A_s\psi_s, \qquad s=1,2,\ldots,m \tag{18b}$$

and $\eta^TC_d^*\boldsymbol{a}_d^*$ and $\eta^{*T}C_d\boldsymbol{a}_d$ represent symbolically the summations

$$\eta^TC_d^*\boldsymbol{a}_d^*=\sum_{s=1}^{m}\sum_{r=s+1}^{m}\eta_{rs}^T[(\psi_s^*\boldsymbol{a}_s^*)_{Ir}-(\psi_r^*\boldsymbol{a}_r^*)_{Is}] \tag{19a}$$

$$\eta^{*T}C_d\boldsymbol{a}_d=\sum_{s=1}^{m}\sum_{r=s+1}^{m}\eta_{rs}^{*T}[(\psi_s\boldsymbol{a}_s)_{Ir}-(\psi_r\boldsymbol{a}_r)_{Is}] \tag{19b}$$

In Eqs. (19), η (η^*) denotes the M-dimensional vector formed by concatenating all of the vectors η_{rs} (η^*_{rs}), and C_d (C_d^*) denotes the $M\times N$ constraint matrix corresponding to the compatibility conditions

$$C_d\boldsymbol{a}_d=\boldsymbol{0} \tag{20a}$$

$$C_d^*\boldsymbol{a}_d^*=\boldsymbol{0} \tag{20b}$$

Conditions (20) respresent M constraints on the disjoint vectors $\boldsymbol{a}_d$ and $\boldsymbol{a}^*_d$. Hence, the assembled reduced order system has only $n=N-M$ independent degrees of freedom. Denoting by $\boldsymbol{a}$ and $\boldsymbol{a}^*$ the n-dimensional independent state vector and costate vector, respectively, the disjoint vectors $\boldsymbol{a}_d$ and $\boldsymbol{a}^*_d$ are related to the vectors $\boldsymbol{a}$ and $\boldsymbol{a}^*$ by the equations

$$\boldsymbol{a}_d(t)=C\boldsymbol{a}(t) \tag{21a}$$

$$\boldsymbol{a}^*_d(t)=C^*\boldsymbol{a}^*(t) \tag{21b}$$

where C and C^* are $N\times n$ constraint matrices. The matrices C and C^* are obtained from Eqs. (20) either by inspection or by

an algorithm based on Householder transformations.[20] Finally, introducing Eqs. (21) into Eq. (17) yields

$$\dot{L} = a^{*T} A \dot{a} + a^{*T} B a \tag{22}$$

in which $A = C^{*T} A_d C$ and $B = C^{*T} B_d C$ are $n \times n$ coefficient matrices for the assembled system. The corresponding assembled system state and costate equation of motion are

$$A\dot{a} + Ba = 0 \tag{23a}$$

$$-A^T \dot{a}^* + B^T a^* = 0 \tag{23b}$$

Equation (23a) can be solved for $a(t)$, and the approximate response in each substructure s $(s=1,2,\ldots,m)$ is given by $y_s(t) = \psi_s a_s(t)$, where $a_s(t)$ is a component of $a_d(t) = Ca(t)$. Of course, similar statements hold for $y_s^*(t)$.

The state equation (23a) describes the evolution of the system states from the initial conditions $a(t_0)$. On the other hand, the costate equations are not of much practical interest except in their role in producing biorthogonal eigenvectors. Considering synchronous motion as in normal mode vibration, the time dependence of Eqs. (23) is eliminated by substituting $a(t) = \alpha \exp(\lambda t)$ and $a^*(t) = \alpha^* \exp(-\lambda t)$. The substitution leads, after some manipulation, to the generalized Rayleigh's quotient[16]

$$R(\lambda) = \lambda = -\alpha^{*T} B \alpha / \alpha^{*T} A \alpha \tag{24}$$

The eigenvectors of the assembled system are then obtained by rendering the quotient stationary, which amounts to solving the eigenproblems

$$A\alpha\Lambda + B\alpha = 0 \tag{25a}$$

$$A^T \alpha^* \Lambda + B^T \alpha^* = 0 \tag{25b}$$

where α is the $n \times n$ matrix of right eigenvectors, α^* the $n \times n$ matrix of left eigenvectors, and Λ the associated $n \times n$ Jordan form.[4,16] When all the eigenvalues are nondegenerate, Λ is a diagonal matrix of computed eigenvalues λ_i $(i=1,2,\ldots,n)$. Accounting for the possibility of degenerate eigenvalues is beyond the scope of this paper. For nondegenerate eigenvalues, the following biorthogonality conditions exist[4,16-18]

$$\alpha^{*T} A \alpha = I \tag{26a}$$

$$\alpha^{*T} B \alpha = \Lambda \tag{26b}$$

In addition, that part of the ith computed right eigenvector which is in substructure s $(s=1,2,\ldots,m)$ is $y_s^{(i)} = \psi_s \alpha_s^{(i)}$, where $\alpha_s^{(i)}$ is a subvector of $\alpha_d^{(i)} = C\alpha^{(i)}$ $(i=1,2,\ldots,n)$. Of course, similar statements hold for the left eigenvector $y_s^{*(i)}$.

IV. Comments on Accuracy and Convergence

The substructure synthesis method is recognized as an application of Galerkin's method to a discrete system composed of m substructures. Trial vectors are defined piecewise over each substructure and the method is a discrete substructure (super-element) counterpart to Galerkin's method applied to distributed-parameter finite elements.[21] An immediate convergence result for the discrete formulation is that an exact model will always be obtained when $N_s = 2n_s$ $(s=1,2,\ldots,m)$ and the substructure trial vectors are all linearly independent. Taking $N_s < 2n_s$ $(s=1,2,\ldots,m)$, however, subjects the system to constraints. When the system is conservative, a bracketing theorem exists which assures that the computed eigenvalues for the system subject to constraints are always no less than the actual system eigenvalues.[4] An analogous bracketing theorem does not exist for general nonconservative systems. The reason is that the generalized Rayleigh's quotient, Eq. (24), is generally not an extremum in the vicinity of an eigenvector; instead, it is merely stationary.[16,17] In the absence of a bracketing theorem, the eigenvalues computed from Eqs. (25) for the assembled system do not converge monotonically with increasing N_s. The eigenvalues are generally complex numbers and those of smallest modulus are the most accurate. However, the modulus of the eigenvalues can converge from above, from below, or by alternating between above and below the actual modulus. In fact, the real part of an eigenvalue can alternate signs as each N_s is increased. Because of these convergence properties, the success of the substructure synthesis method depends on the ability to produced a highly accurate computed eigensolution while using small numbers N_s of trial vectors (see Sec. VI).

V. Selection of Substructure Trial Vectors

May possible choices of substructure trial vectors ψ_{si} and ψ_{si}^* $(i=1,2,\ldots,N_s)$ exist. References 12-14 advocate using a number of substructure right and left eigenvectors as the trial vectors ψ_{si} and ψ_{si}^*, respectively, where the eigenvectors are obtained by solving an eigenproblem for the right eigenvectors and an adjoint eigenproblem for the left eigenvectors. However, using substructure eigenvectors is not advantageous for the following main reasons: 1) the associated computational burden is high, 2) the eigenvectors are generally complex quantities and their use leads to complex assembled system matrices A and B, and 3) a great deal of ambiguity exists concerning the boundary conditions to be imposed at internal boundaries when defining substructure eigenproblems, i.e., should the internal boundary points be fixed, should they be free, or should they be loaded in some fashion. This paper, on the other hand, proposes using a number of real trial vectors ψ_{si} and ψ_{si}^* that have no relationship to a *substructure* eigenproblem. Whereas the trial vectors are real, their coefficients, i.e., the entries of α_s and α_s^*, are permitted to be complex quantities. The interest is in producing a suitable set of N_s trial vectors for each substructure s with the minimal amount of effort. The number of possible choices is progressively narrowed in the following discussion.

First, because the substructure synthesis method is a Galerkin method, it is always permissible to choose $\psi_{si} = \psi_{si}^*$ $(i=1,2,\ldots,N_s)$, i.e., to choose $\psi_s = \psi_s^*$. Then, Galerkin's method realizes a projection[21] of each substructure state and costate vector onto a subspace spanned by the N_s columns of ψ_s $(s=1,2,\ldots,m)$. Choosing $\psi_s = \psi_s^*$ reduces the computer storage and bookkeeping requirements associated with the synthesis. The choice also preserves any symmetry and/or skew symmetry inherent in the substructure matrices A_s and B_s. The sequel assumes $\psi_s = \psi_s^*$ $(s=1,2,\ldots,m)$.

Next, consider the following partitioned form for the $2n_s \times N_s$ matrices ψ_s and ψ_s^*

$$\psi_s = \psi_s^* = \begin{bmatrix} \phi_{sd} & 0 \\ 0 & \phi_{sv} \end{bmatrix}, \qquad s=1,2,\ldots,m \tag{27}$$

where ϕ_{sd} is an $n_s \times N_{sd}$ matrix, ϕ_{sv} is an $n_s \times N_{sv}$ matrix, and $N_{sd} + N_{sv} = N_s$. The choice of Eq. (27) is equivalent to representing the displacement vector u_s independently of the velocity vector v_s, where a linear combination of the N_{sd} columns of ϕ_{sd} describes u_s and a linear combination of the N_{sv} columns of ϕ_{sv} describes v_s. The independent representation of the displacement and the velocity for distributed conservative systems is discussed in detail in Ref. 19. Here, the columns of ϕ_{sd} and ϕ_{sv} are chosen as substructure admissible vectors for the displacement u_s and the velocity v_s, respectively. Admissible vectors represent the discrete analogue of the concept of admissible functions for distributed systems (see Refs. 11 and 15). For a discrete substructure, they are obtained by first choosing a set of admissible functions for a dynamically similar distributed substructure. Then, appropriate entries of an admissible

vector are selected as the values of an admissible function at nodal points of the substructure, and other appropriate entries are selected as the values of the function's spatial derivatives. The further simplification $\phi_{sd}=\phi_{sv}$ ($N_s=2N_{sd}=2N_{sv}$) is also permissible. In fact, it is common practice to choose the columns of ϕ_{sd} (also ϕ_{sv}) as a number of the eigenvectors obtained from solving an associated conservative nongyroscopic substructure eigenproblem. Although the eigenvectors are real, the fact that they are eigenvectors is unnecesary and other substructure admissible vectors suffice.

Associated with an independent representation of $\boldsymbol{u}_s$ and $\boldsymbol{v}_s$ is the intuitively appealing idea that the compatibility conditions $(\boldsymbol{v}_s)_{Ir}-(\boldsymbol{v}_r)_{Is}=\boldsymbol{0}$ and $(\boldsymbol{w}_s)_{Ir}-(\boldsymbol{w}_r)_{Is}=\boldsymbol{0}$ should be satisfied, either exactly or approximately, in addition to the conditions $(\boldsymbol{u}_s)_{Ir}-(\boldsymbol{u}_r)_{Is}=\boldsymbol{0}$ and $(\boldsymbol{u}_s^*)_{Ir}-(\boldsymbol{u}_r^*)_{Is}=\boldsymbol{0}$. Only the latter conditions are required to be satisfied as far as the vartiational derivation of the substructure state and costate equations is concerned. Upon close scrutiny of Sec. III, one realizes that imposing only the conditions of Eqs. (20) leads to different matrices C and C^* in Eqs. (21), even when $\psi_s=\psi_s^*$ ($s=1,2,\ldots,m$). Any symmetry and/or skew symmetry inherent in the substructure matrices A_s and B_s that is preserved by choosing $\psi_s=\psi_s^*$ is then destroyed by the coupling process. In fact, both assembled system matrices A and B are asymmetric. Moreover, the coupling process so far can lead to a singular matrix A, where the singularity can cause computational difficulties. The entries in the vectors $\boldsymbol{y}_s$ and $\boldsymbol{y}_s^*$ associated with the components of $\boldsymbol{v}_s$ and $\boldsymbol{w}_s$, respectively, at the internal boundary points $P\epsilon S_{rs}$ can be denoted by $(\,)_{\bar{I}r}$. Then, requiring $(\boldsymbol{v}_s)_{Ir}-(\boldsymbol{v}_r)_{Is}=\boldsymbol{0}$ and $(\boldsymbol{w}_s)_{Ir}-(\boldsymbol{w}_r)_{Is}=\boldsymbol{0}$ to be satisfied can be written symbolically as

$$(\boldsymbol{y}_s)_{\bar{I}r}-(\boldsymbol{y}_r)_{\bar{I}s}=\boldsymbol{0} \tag{28a}$$

$$(\boldsymbol{y}_s^*)_{\bar{I}r}-(\boldsymbol{y}_r^*)_{\bar{I}s}=\boldsymbol{0} \tag{28b}$$

Equations (28) lead to constraints that compliment Eqs. (15) in such a way that $C=C^*$ in Eqs. (21), where now $n=N-2M$. The sequel considers that the conditions of Eqs. (15) as well as Eqs. (28) are reflected in matrices $C=C^*$ in Eqs. (21).

VI. Subspace Iteration Based on Substructure Synthesis

One measure of the accuracy of the assembled system model is how closely the eigenvalues of smallest modulus and associated eigenvectors computed from Eqs. (25) represent the actual system eigenvalues and associated eigenvectors. For single structures described by asymmetric matrices, Ref. 7 developed a method similar to subspace iteration to iteratively improve the accuracy of the computed eigensolution without increasing the number of degrees of freedom. Assuming that q nondegenerate eigenvalues of smallest modulus are desired, q real trial vectors are selected initially. Then, one step of a subspace iteration method as modified from Ref. 7 consists of the following three steps: 1) represent the actual system via Galerkin's method using the latest trial vectors, and compute the approximate eigenvalues and associated right eigenvectors; 2) stop if the computed eigensolution is of the desired accuracy; otherwise 3) perform at least one matrix iteration simultaneously on the q_{Re} real computed eigenvectors, on the q_c *real parts* of the q_c complex conjugate pairs of eigenvectors and also on the associated q_c *imaginary parts,* where $q=q_{\mathrm{Re}}+2q_c$, and q_{Re} and q_c denote the number of real and the number of complex conjugate pairs of eigenvalues, respectively; use the resulting q *real vectors* as improved trial vectors in step 1. The method converges to the q actual system eigenvalues of smallest modulus and the associated q eigenvectors, where convergence occurs first for the smallest eigenvalues. Note that the left eigenvectors need not be computed in this method, in contrast to Refs. 5 and 6. Moreover, only one simultaneous matrix iteration is necessary in step 3 although two or more iterations can be performed before repeating step 1.

By analogy with steps 1-3 for a single structure, a subspace iteration method exists for systems composed of substructures. The idea is to iteratively improve substructure trial vectors without increasing their number N_s. The method, as considered here, consists of the following three steps: 1s) represent the actual system via the substructure synthesis method of Secs. II and III using the latest substructure trial vectors, and compute the approximate eigenvalues and associated right eigenvectors from Eq. (25a); 2s) stop if the computed eigensolution is of the desired accuracy; otherwise 3s) for each substructure individually, perform at least one substructure matrix iteration simultaneously on q_{Re} real computed eigenvectors, on the q_c *real parts* of q_c complex conjugate pairs of computed eigenvectors, and also on the associated q_c *imaginary parts*, where the eigenvectors are those for the $q=q_{\mathrm{Re}}+2q_c$ computed eigenvalues of smallest modulus; use the resulting *real* substructure vectors in step 1s.

Whereas the matrix iteration in step 3 is based on a "reciprocal" formulation for the whole system eigenproblem, the substructure matrix iteration in step 3s is based on an analogus reciprical formulation for each substructure and on a corresponding specific choice of substructure trial vectors. The reciprocal formulation of a generic substructure is now discussed briefly.

The whole system eigenproblem and its adjoint as defined in substructure s ($s=1,2,\ldots,m$) are described by the algebraic equations

$$\lambda A_s\boldsymbol{y}_s+B_s\boldsymbol{y}_s=\boldsymbol{\gamma}_s \tag{29a}$$

$$\lambda A_s\boldsymbol{y}_s^*+B_s^T\boldsymbol{y}_s^*=\boldsymbol{\gamma}_s^* \tag{29b}$$

Equations (29) are the result of eliminating the time dependence of Eq. (11). Assuming known vectors $\boldsymbol{\gamma}_s$ and $\boldsymbol{\gamma}_s^*$, the unique solution to Eqs. (29) can be written in the general forms

$$\boldsymbol{y}_s=\boldsymbol{f}_s+\lambda\mathring{A}_s\boldsymbol{y}_s \tag{30a}$$

$$\boldsymbol{y}_s^*=\boldsymbol{f}_s^*+\lambda\mathring{A}_s^*\boldsymbol{y}_s^* \tag{30b}$$

where for a nonsingular matrix B_s, and in view of Eqs. (13),

$$\mathring{A}_s=-B_s^{-1}A_s \tag{31a}$$

$$\mathring{A}_s^*=-B^{-T}A_s \tag{31b}$$

$$\boldsymbol{f}_s=B_s^{-1}\boldsymbol{\gamma}_s=\sum_{\substack{r=1\\r\neq s}}^{m}(B_s^{-1})_{Ir}\boldsymbol{\eta}_{rs}=\sum_{\substack{r=1\\r\neq s}}^{m}\sum_{i=1}^{m_{rs}}(B_s^{-1})_{Ir}\boldsymbol{g}_{rs,i}a_{rsi}=F_{sc}\boldsymbol{a}_{sc} \tag{32a}$$

$$\boldsymbol{f}_s^*=B_s^{-T}\boldsymbol{\gamma}_s^*=\sum_{\substack{r=1\\r\neq s}}^{m}(B_s^{-T})_{Ir}\boldsymbol{\eta}_{rs}^*=\sum_{\substack{r=1\\r\neq s}}^{m}\sum_{i=1}^{m_{rs}}(B_s^{-T})_{Ir}\boldsymbol{g}_{rs,i}a_{rsi}^*=F_{sc}^*\boldsymbol{a}_{sc}^* \tag{32b}$$

In Eqs. (32), $(B_s^{-1})_{Ir}$ and $(B_s^{-T})_{Ir}$ denote $2n_s\times m_{rs}$ matrices formed by the columns of B_s^{-1} and B_s^{-T} that multiply the components of $\boldsymbol{u}_s$ and $\boldsymbol{u}_s^*$, respectively, corresponding to the internal boundary points $P\epsilon S_{rs}$. The m_{rs} dimensional unit vectors $\boldsymbol{g}_{rs,i}$ represent the independent basis vectors of a series expansion of $\boldsymbol{\eta}_{rs}$ and $\boldsymbol{\eta}_{rs}^*$, where the coefficients a_{rsi} and a_{rsi}^*, respectively, are determined by the satisfaction of the compatibility conditions of Eqs. (15). Moreover, F_{sc} (F_{sc}^*) is a $2n_s\times M_{sc}$ matrix, $\boldsymbol{a}_{sc}$ ($\boldsymbol{a}_{sc}^*$) is an M_{sc}-dimensional coefficients vector, and

$$M_{sc}=\sum_{\substack{r=1\\s\neq s}}^{m}m_{rs}$$

Note that Eqs. (30) are reciprocal formulations of Eqs. (29) and that for known vectors f_s and f_s^* they can be solved iteratively using the formulas

$$y_s^{p+1} = f_s + \lambda \mathring{A}_s y_s^p \tag{33a}$$

$$y_s^{*p+1} = f_s^* + \lambda \mathring{A}_s y_s^{*p} \tag{33b}$$

If k_{cs} in Eq. (1) is semidefinite, i.e., singular, the possibility exists that k_s in Eqs. (3) and (4) is also singular. The singularity arises because the substructure acting alone is capable of rigid body motions. The singularity of k_s leads to a singular matrix B_s and Eqs. (31) and (32) must be modified. To this end, one artificially imposes the independent constraints

$$C_s u_s = 0 \tag{34a}$$

$$C_s^* u_s^* = 0 \tag{34b}$$

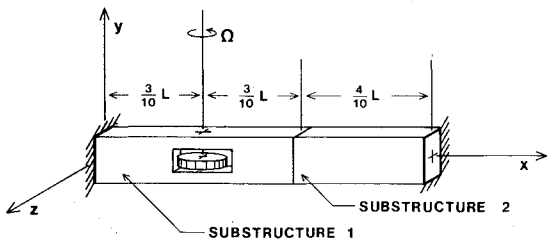

Fig. 1 Beam with momentum wheel.

where C_s (C_s^*) is an $n_{Rs} \times n_s$ constraint matrix and n_{Rs} is the number of substructure rigid body modes, i.e., displacements (adjoint displacements) satisfying $k_s u_s = 0$ ($k_s^T u_s^* = 0$). The constraints of Eqs. (34) are then used to relate a $(2n_s - n_{Rs})$ dimensional vector $\bar{y}_s$ ($\bar{y}_s^*$) of independent states (costates) to a $2n_s$-dimensional constrained state (costate) vector y_{sc} (y_{sc}^*), i.e., to write $y_{sc} = C_{sc}\bar{y}_s$ ($y_{sc}^* = C_{sc}^*\bar{y}_s^*$) where C_{sc} (C_{sc}^*) is a $2n_s \times (2n_s - n_{Rs})$ matrix. Finally, solutions of Eqs. (29) in the form of Eqs. (30) are obtained in which B_s^{-1} is replaced by $\theta_s = C_{sc}(C_{sc}^{*T} B_s C_{sc})^{-1} C_{sc}^{*T}$ (B_s^{-T} by θ_s^T) and the first n_{Rs} columns of F_{sc} (F_{sc}^*) are taken as $y_s^{(i)} = \{u_s^{(i)T}, 0^T\}^T$ ($y_s^{*(i)} = \{0^T, u_s^{(i)T}\}^T$) in which $u_s^{(i)}$ are the rigid body modes for substructure s.

Based on Eqs. (33), different subspace iteration algorithms (steps 1s-3s) can now be derived depending on the procedure used to accommodate the vectors f_s and f_s^* which are actually unknown. In the case of an assembled structure, f_s (f_s^*) contains the unknown coefficients vector a_{sc} (a_{sc}^*). In a subspace iteration algorithm, the coefficients a_{sc} (a_{sc}^*) for each substructure at iteration $p+1$ must be determined so as to ensure that y_s^{p+1} (y_s^{*p+1}) ($s = 1,2,\ldots,m$) satisfies the compatibility conditions of Eq. (15a) [Eq. (15b)]. For brevity, only one convenient algorithm is discussed in the sequel. A rigorous proof of its convergence is omitted also for brevity.

The focus is on obtaining the $q = q_{Re} + 2q_c$ nonzero eigenvalues of smallest modulus and the associated eigenvectors by an algorithm in which the coefficients a_{sc} (a_{sc}^*) are *permitted to be evaluated automatically* in the substructure synthesis of step 1s. A total of n_R rigid body modes are assumed to exist, where n_R can equal zero. For a_{sc} (a_{sc}^*) ($s = 1,2,\ldots,m$) to be determined by the synthesis, however, the columns of F_{sc} and F_{sc}^* must first be imbedded in the

Table 1 Beam computed eigenvalues ($\Omega = 0$)

No. of the computed eigenvalue	Initial synthesis ($p=0$)	First iteration ($p=1$)	Second iteration ($p=2$)	Third iteration ($p=3$)	"Exact" eigenvalues
1	−0.0006227 +i0.2065096	−0.0006219 +i0.2064217	−0.0006219 +i0.2064217	−0.0006219 +i0.2064217	−0.0006219 +i0.2064217
2	−0.0058473 +i0.6244758	−0.0051725 +i0.6025639	−0.0051706 +i0.6025246	−0.0051706 +i0.6025246	−0.0051706 +i0.6025246
3	−0.0236411 +i1.1988642	−0.0208031 +i1.1091699	−0.0207435 +i1.1071998	−0.0207369 +i1.1071683	−0.0207367 +i1.1071666
4	−0.0796903 +i2.2103517	−0.0392866 +i1.7821442	−0.0403278 +i2.0962537	−0.0575367 +i2.1470834	−0.0303895 +i1.7279849
5	−0.2318182 +i3.2947460	−0.1062026 +i2.4038095	−0.0751689 +i2.5746271	−0.0827553 +i2.6679982	−0.0620151 +i2.0886966

Table 2 Beam computed eigenvalues ($\Omega = 10$)

No. of the computed eigenvalue	Initial synthesis ($p=0$)	First iteration ($p=1$)	Second iteration ($p=2$)	Third iteration ($p=3$)	"Exact" eigenvalues
1	−0.0006239 +i0.2174061	−0.0006231 +i0.2172970	−0.0006231 +i0.2172970	−0.0006231 +i0.2172970	−0.0006231 +i0.2172970
2	−0.0146428 +i0.8668180	−0.0144939 +i0.8639635	−0.0144948 +i0.8639628	−0.0144948 +i0.8639628	−0.0144948 +i0.8639628
3	−0.0261544 +i1.5527242	−0.0241871 +i1.5162218	−0.0243780 +i1.5157982	−0.0243784 +i1.5157865	−0.0243781 +i1.5157849
4	−0.0976934 +i2.3168847	−0.0872216 +i2.2152751	−0.0869722 +i2.2159463	−0.0870488 +i2.2159404	−0.0867883 +i2.2152245
5	−0.5034160 +i4.7958659	−0.3833706 +i4.3174956	−0.3794548 +i4.3281201	−0.3769083 +i4.3246927	−0.1404752 +i3.9334985

representation of each substructure s. To this end, the substructure trial vectors can be chosen so that at iteration p

$$y_{s,p} = [F_{sc} \quad F^*_{sc} \quad \bar{\psi}^p_s]\, a_s = \psi_s \alpha_s \tag{35a}$$

$$y^*_{s,p} = [F_{sc} \quad F^*_{sc} \quad \bar{\psi}^p_s]\, a^*_s = \psi_s \alpha^*_s \tag{35b}$$

Note that the columns of F_{sc} and F^*_{sc} are always linearly independent and that, because it is desired for simplicity to use $\psi_s = \psi^*_s$, both F_{sc} and F^*_{sc} are included in each of Eqs. (35). The choice of Eqs. (35) also expedites obtaining the $N \times n$ constraint matrices $C = C^*$. In Eqs. (35), ψ_s (ψ^*_s) is a $2n_s \times N_s$ matrix, where $N_s = 2M_{sc} + q$. The last q columns of ψ_s (ψ^*_s) when $p = 0$, i.e., $\bar{\psi}^0_s$, are taken to be substructure trial state vectors that are independent of the columns of F_{sc} and F^*_{sc}. The number q is taken to be the same for all substructures.

At each iteration $p = 0,1,\ldots$, an approximate eigensolution is computed during step 1s by solving the algebraic eigenproblem of Eq. (25a). The eigenproblem is obtained by the substructure synthesis method in which each substructure is represented by the sums of Eq. (35). The algebraic eigensolution consists of n computed eigenvalues $\lambda_{(i)}$ ($i = 1,2,\ldots,n$) approximating the true eigenvalues. Corresponding to each eigenvalue $\lambda_{(i)}$ are values of the coefficients vectors $\alpha^{(i)}_s$ which, then substituted into Eq. (35a) for each substructure s, yield that part of the computed right eigenvector within substructure s, namely, $y^{(i)}_{s,p}$. The interest is in generating substructure trial vectors that better represent the eigenvectors associated with the q nonzero computed eigenvalues of smallest modulus. To this end, the computed eigenvalue-eigenvector pairs are ordered so that $\lambda_{(i)} = 0$ ($i = 1,2,\ldots,n_R$), $\lambda_{(n_R+i)}$ ($i = 1,2,\ldots,q_{\mathrm{Re}}$) are real eigenvalues, and $\lambda_{(n_R+q_{\mathrm{Re}}+i)}$ ($i = 1,2,\ldots,q_c$) are complex eigenvalues with the conjugate values $\lambda_{(n_R+q_{\mathrm{Re}}+q_c+i)}$. The latter ordering of the complex eigenvalues is crucial to an understanding of the following equations. Based on Eq. (33a) for each substructure s, step 3s computes q improved trial vectors $\bar{\psi}^{p+1}_s = [\bar{\psi}^{p+1}_{s,1}, \ldots, \bar{\psi}^{p+1}_{s,q}]$ for use at iteration $p+1$ in Eqs. (35), where

$$\bar{\psi}^{p+1}_{s,i} = \mathring{A}_s y^{(n_R+i)}_{s,p}, \qquad i = 1,2,\ldots,q_{\mathrm{Re}} \tag{36a}$$

$$\bar{\psi}^{p+1}_{s,i} = \mathring{A}_s (y^{(n_R+i)}_{s,p})_{\mathrm{Re}}, \qquad i = q_{\mathrm{Re}}+1,\ldots,q_{\mathrm{Re}}+q_c \tag{36b}$$

$$\bar{\psi}^{p+1}_{s,q_c+i} = \mathring{A}_s (y^{(n_R+i)}_{s,p})_{\mathrm{Im}}, \qquad i = q_{\mathrm{Re}}+1,\ldots,q_{\mathrm{Re}}+q_c \tag{36c}$$

Note that $(\ )_{\mathrm{Re}}$ denotes the real part and $(\ )_{\mathrm{Im}}$ denotes the imaginary part of an eigenvector. Equations (36) constitute one step of the substructure matrix iteration although here the iteration is on computed eigenvectors for the *assembled system*. The iteration *does not* produce substructure eigenvectors. The computations indicated by Eqs. (36) are independent of all other substructures and they can be performed in parallel. The method always yields the n_R rigid body modes, and it can be shown to converge to the q nonzero true eigenvalues of smallest modulus and their associated eigenvectors. However, the substructure synthesis method always produces $n > n_R + q$ computed eigenvalues, and the other $n - n_R - q$ eigenvalues do not converge to true eigenvalues. Although some of them can be accurate approximations to true eigenvalues, others of them can be grossly inaccurate and one should not base any stability conclusions on the higher $n - n_R - q$ computed eigenvalues.

The matrix operator $\mathring{A}_s$ in Eqs. (36) is defined in terms of the inverse of the $2n_s \times 2n_s$ matrix B_s. In practice, it is not computationally expedient to invert B_s and herein $\bar{\psi}^{p+1}_{s,i} = \{(\bar{\psi}^{p+1}_{sa,i})^T, (\bar{\psi}^{p+1}_{sb,i})^T\}^T$ ($i = 1,2,\ldots,q_{\mathrm{Re}}$) are the solutions to[7]

$$k_s \bar{\psi}^{p+1}_{sa,i} = -m_s v^{(n_R+i)}_{s,p} - c_s u^{(n_R+i)}_{s,p} \tag{37a}$$

$$\bar{\psi}^{p+1}_{sb,i} = u^{(n_R+i)}_{s,p}, \qquad i = 1,2,\ldots,q_{\mathrm{Re}} \tag{37b}$$

in which $y^{(k)}_{s,p} = \{u^{(k)T}_{s,p}, v^{(k)T}_{s,p}\}^T$. Equations (37) replace Eq. (36a), and analogous equations replace Eqs. (36b) and (36c). According to Eqs. (37) only n_s simultaneous equations need actually be solved for each substructure s. However, if a poor choice of trial vectors yields inaccurate $u^{(n_R+i)}_{s,p}$ ($i = 1,2,\ldots,q$), Eqs. (37) also suggest that at least two subspace iterations must be performed before the $u^{(n_R+i)}_{s,p}$ propagate out of the improved trial vectors. The situation can be alleviated for only a small additional effort at each iteration p by replacing Eqs. (37) with

$$k_s \bar{\psi}^{p+1}_{sb,i} = -m_s v^{(n_R+i)}_{s,p} - c_s u^{(n_R+i)}_{s,p}, \qquad i = 1,2,\ldots,q_{\mathrm{Re}} \tag{38a}$$

$$k_s \bar{\psi}^{p+1}_{sa,i} = -m_s v^{(n_R+i)}_{s,p} - c_s \bar{\psi}^{p+1}_{sb,i}, \qquad i = 1,2,\ldots,q_{\mathrm{Re}} \tag{38b}$$

Equations (38) are equivalent to performing two substructure matrix iterations in step 3s of iteration p and they correspond to replacing Eq. (33a) with

$$y^{p+1}_s = f_s + \lambda \mathring{A}_s f_s + \lambda^2 \mathring{A}^2_s y^p_s \tag{39}$$

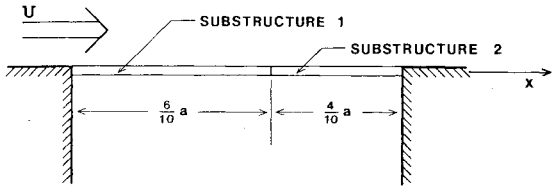

Fig. 2 Plate in supersonic flow.

Table 3 Plate computed eigenvalues ($\alpha = 5$)

No. of the computed eigenvalue	Initial synthesis ($p = 0$)	First iteration ($p = 1$)	Second iteration ($p = 2$)	Third iteration ($p = 3$)	"Exact" eigenvalues
1	0.0890539 + i0.3921328	0.0887682 + i0.3937127	0.0887462 + i0.3937473	0.0887460 + i0.3937492	0.0887460 + i0.3937493
2	−0.1030583 + i0.3920984	−0.1029046 + i0.3937094	−0.1028879 + i0.3937473	−0.1028881 + i0.3937493	−0.1028882 + i0.3937493
3	−0.0071809 + i0.7822793	−0.0070346 + i0.7844499	−0.0070566 + i0.7865089	−0.0070670 + i0.7869172	−0.0070711 + i0.7870096
4	−0.0097276 + i1.3000696	−0.0072270 + i1.2191996	−0.0071913 + i1.2217305	−0.0071982 + i1.2222081	−0.0070711 + i1.2025250
5	−0.0033289 + i1.6141749	−0.0075647 + i1.4594073	−0.0074851 + i1.4663935	−0.0075087 + i1.4677552	−0.0070711 + i1.3974462

This alternative can be implemented without modification to Eqs. (35) because the first two terms in Eq. (39) can always be produced by a linear combination of the M_{sc} columns of F_{sc} and the M_{sc} columns of F^*_{sc}. The appropriate linear combination is determined automatically in the synthesis of step 1s at iteration $p+1$. Any linear combination of the columns of F_{sc} and F^*_{sc} can also be represented by an appropriate linear combination of the columns of F_{sc} and $\hat{A}_s F_{sc}$, and F^*_{sc} in Eqs. (35) can be replaced by $A_s F_{sc}$, a matrix which is more expedient to compute than F^*_{sc}. The algorithm that produced the example results (Sec. VII) uses $A_s F_{sc}$ as well as Eqs. (38).

VII. Illustrative Examples

The substructure synthesis method (Secs. II and III) and the iterative improvement procedure (Sec. VI) are developed specifically for large structural systems. However, it is convenient to demonstrate possible applications via two very simple examples.

The first example consists of the combined transverse and torsional vibration analysis of a fixed-fixed discontinuous beam with an embedded momentum wheel (Fig. 1). The beam is divided into two substructures and, for the sake of example, substructure one (two) is divided into six (four) finite elements having length h. The bending stiffness EI_s and torsional stiffness GJ_s $(s=1,2)$ are $EI_2=2EI_1=2EI$ and $GJ_2=2GJ_1=40EI/h^2$, respectively. The mass per unit length ρ is constant and the constant polar moment of inertia is $J_m=0.1\rho$. The momentum wheel's mass M_w and moments of inertial I_x and I_z are taken as $M_w=I_x=I_z=2\rho h$. Both substructures are assumed to possess viscous damping proportional to the stiffness, where $c_{v1}=k_{c1}/50$ and $c_{v2}=k_{c2}/20$. Therefore, the assembled system is not only gyroscopic but it is nonconservative. Because each substructure is capable of transverse displacements in the y direction and torsion, there are three degrees of freedom per finite element nodal point. Hence, $n_1=18$, $n_2=12$, and $m_{12}=3$. Starting from a choice of $q=2q_c=6$ admissible vectors $\bar{\psi}^0_{s,i}$ $(i=1,2,\ldots,q;\ s=1,2)$, the substructure synthesis method and the subspace iteration algorithm are used to iteratively produce the $q=6$ eigenvalues of smallest modulus and associated eigenvectors for the assembled system, where always $N_1=N_2=12$. For $\Omega=0$, the computed eigenvalues of smallest modulus $\lambda_{(i)}/(EI/\rho h^4)^{1/2}$ $(i=1,2,3,4,5)$ are displayed in Table 1 for iterations $p=0,1,2$, and 3. For each eigenvalue displayed a complex conjugate eigenvalue is also computed. Similar results for $\Omega=10$ are displayed in Table 2. The actual eigenvalues of the finite element model are displayed in the last column of Tables 1 and 2 for comparison. Because $q_c=3$, only the first three pairs of eigenvalues converge to true eigenvalues although here the remaining eigenvalues approximate the true eigenvalues. Note that the convergence is rapid. The stiffening due to the momentum wheel is also readily observed when comparing Tables 1 and 2.

The second example consists of a discontinuous elastic plate embedded in a rigid surface (Fig. 2). A fluid flows at a high supersonic speed U over the top of the plate. The plate is divided into two substructures, where the nondimensionalized equation of motion for each substructure s is (see Ref. 14).

$$\lambda^2 w_s+\lambda(\mu/\alpha M^2)^{1/2} w_s+\epsilon_s w_s''''+(\alpha/M) w_s'=0,\quad s=1,2 \qquad (40)$$

where w_s is the deflection, M the Mach number, $\mu=\rho a/m$, $\alpha=\rho U^2 a^3/D$, $\epsilon_s D$ is the stiffness of substructure s, m the constant mass per unit length, and ρ the flow density. Herein, $\epsilon_1=1$, $\epsilon_2=2$, and $a=15$. Equations (40) are subject to appropriate boundary conditions and they can be discretized by the finite element method. For the sake of example, substructure one (two) is divided into nine (six) finite elements of length $h=a/15$. There are two degrees of freedom per nodal point so that $q=2q_c=6$ admissible vectors $\bar{\psi}^0_{s,i}$ are chosen initially. The $q=6$ eigenvalues of smallest modulus are iteratively produced, where always $N_1=N_2=10$. For $\alpha=5$, $M=10$, and $\mu=0.1$, the computed eigenvalues $\lambda_{(i)}$ $(i=1,2,3,4,5)$ are displayed in Table 3 for $p=0,1,2$, and 3. Note that the system is unstable, i.e., flutter occurs. Nevertheless, the method produces the first q_c pairs of true system eigenvalues.

VIII. Concluding Remarks

This paper has presented a general substructure synthesis method for the dynamic analysis of nonconservative vibratory systems. The synthesis method can be used as an integral step of an algorithm for iteratively improving the representation of each substructure. The improvement algorithm permits obtaining an actual partial eigensolution. At each iteration, the calculations for each substructure are independent of those for all other substructures and they can be performed in parallel. The methodology is applicable to many large nonconservative linear systems.

References

[1] Parlett, B. N., *The Symmetric Eigenvalue Problem*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1980.

[2] Bathe, K. J. and Wilson, L., *Numerical Methods in Finite Element Analysis*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1976.

[3] Jennings, A., *Matrix Computation for Engineers and Scientists*, John Wiley, N. Y., 1977.

[4] Meirovitch, L., *Computational Methods in Structural Dynamics*, Sijthoff-Noordhoff International Publishers, Alphen an den Rijin, The Netherlands, 1980.

[5] Clint, M. and Jennings, A., "A Simultaneous Iteration Method for the Unsymmetric Eigenvalue Problem," *Journal of the Institute of Mathematics and Its Applications*, Vol. 8, 1971, pp. 111-121.

[6] Jennings, A. and Stewart, W. J., "Simultaneous Iteration for the Partial Eigensolution of Real Matrices," *Journal of the Institute of Mathematics and Its Applications*, Vol. 15, 1975, pp. 351-361.

[7] Dong, S. B., "A Block-Stodola Eigensolution Technique for Large Algebraic Systems with Non-Symmetrical Matrices," *International Journal for Numerical Methods in Engineering*, Vol. 11, 1977, pp. 247-267.

[8] Hurty, W. C., "Dynamic Analysis of Structural Systems Using Component Modes," *AIAA Journal*, Vol. 3, 1965, pp. 678-685.

[9] Gladwell, G. M. L., "Branch Mode Analysis of Vibrating Systems," *Journal of Sound and Vibration*, Vol. 1, 1964, pp. 41-59.

[10] Craig, R. R., "Methods of Component Modes Synthesis," *The Shock and Vibration Digest*, Vol. 9, Nov. 1977, pp. 3-10.

[11] Meirovitch, L. and Hale, A. L., "On the Substructure Synthesis Method," *AIAA Journal*, Vol. 19, July 1981, pp. 940-947.

[12] Hasselman, T. K. and Kaplan, A., "Dynamic Analysis of Large Systems by Complex Mode Substitution," *Journal of Dynamic Systems, Measurement and Control, Transactions of the ASME*, Vol. 96, Ser. G., Sept. 1974, pp. 327-333.

[13] Craig, R. R. Jr. and Chung, Y-T, "A Generalized Substructure Coupling Procedure for Damped Systems," Proceedings of the 22nd AIAA Structures, Structural Dynamics, and Materials Conference, Atlanta, Ga., April 6-8, 1981.

[14] Dowell, E. H., "Component Mode Analysis of Nonlinear and Nonconservative Systems," *Journal of Applied Mechanics, Transactions of the ASME*, Vol. 47, 1980, pp. 172-176.

[15] Hale, A. L. and Meirovitch, L., "A Procedure for Improving Discrete Substructures Representation in Dynamic Synthesis," *AIAA Journal*, Vol. 20, Aug. 1982, pp. 1128-1136.

[16] Huseyin, K., *Vibrations and Stability of Multiple Parameter Systems*, Sijthoff-Noordhoff International Publishers, Alphen ann den Rijin, The Netherlands, 1978.

[17] Leipholz, H., *Direct Variational Methods and Eigenvalue Problems in Engineering*, Noordhoff International Publishers, Leyden, The Netherlands, 1977.

[18] Meirovitch, L., *Analytical Methods in Vibrations*, The MacMillian Co., New York, 1967.

[19] Hughes, T. J. R., Hilber, H. M., and Taylor, R. L., "A Reduction Scheme for Problems of Structural Dynamics," *International Journal of Solids and Structures*, Vol. 12, 1976, pp. 749-767.

[20] Golub, H. G. and Underwood, R., "Stationary Values of the Ratio of Quadratic Forms Subject to Linear Constraints," *Zeitschrift fur Angewandte Mathematik and Physik*, Vol. 21, 1970, pp. 318-326.

[21] Oden, J. T. and Reddy, J. N., *An Introduction to the Mathematical Theory of Finite Elements*, John Wiley and Sons, New York, 1976.